A Proof that 0/0=2

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A common joke on mathematics forums and Facebook groups is to claim that

$$\frac{0}{0} = 2$$

The "proof" usually proceeds as a chain of equalities:

$$\frac{0}{0} = \frac{1-1}{1-1} = \frac{1^2-1^2}{1-1} = \frac{(1-1)(1+1)}{1-1} = 1+1 = 2$$

Of course such a proof is invalid, because the denominator of these fractions is 0 and division by 0 is an invalid mathematical operation. This document presents an analysis "proof" whose mathematical flaw is much more subtle.

We define a function $f(x)$ thus:

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

We note that $f(x)$ is uniformly continuous over its entire domain and is therefore continuous over its entire domain. A function is uniformly continuous over an interval I if for any real $\epsilon > 0$, there exists a real $\delta < 0$ such that if $x_1, x_2 \in I$ and $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \epsilon$. This can easily be verified by setting $\epsilon = 2\delta$:

$$\begin{aligned} |x_1 - x_2| &< \delta \\ |f(x_1) - f(x_2)| &= \left| \frac{(x_1)^2 - 1}{x_1 - 1} - \frac{(x_2)^2 - 1}{x_2 - 1} \right| \\ &= |x_1 - x_2| \\ &< 2\delta \\ &< \epsilon \end{aligned}$$

The denominators can be canceled as long as $x_1 \neq 1$ and $x_2 \neq 1$. If either value is unit, however, we must be more careful. We can assume without loss of generality that $x_1 = 1$:

$$\begin{aligned} |1 - x_2| &< \delta \\ |f(x_1) - f(x_2)| &= \left| 2 - \frac{(x_2)^2 - 1}{x_2 - 1} \right| \\ &= |1 - x_2| \\ &< 2\delta \\ &< \epsilon \end{aligned}$$

Our last case $x_1 = x_2 = 1$ is trivial, because $|x_1 - x_2| = |f(x_1) - f(x_2)| = 0$.

We now evaluate the limit of $f(x)$ as x approaches 1. We don't want to divide by 0, so we use the limit quotient rule to help us. The quotient rule is well-established and can be found in any calculus text.

$$\begin{aligned}
\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2-1}{x-1} \\
&= \frac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x-1)} \\
&= \frac{1-1}{1-1} \\
&= \frac{0}{0}
\end{aligned}$$

By the definition of continuity, for any c in the domain:

$$\lim_{x\to c} f(x) = f(c)$$

Setting $c = 1$:

$$\begin{aligned}
\lim_{x\to 1} f(x) &= f(1) \\
\frac{0}{0} &= 2
\end{aligned}$$

What went wrong here? $f(x)$ is indeed uniformly continuous; the quotient rule for limits is indeed a valid rule; and a continuous function at a value does indeed equal its limit as it approaches that value. All the rules of analysis were applied correctly. So why do we arrive at the conclusion that $\frac{0}{0} = 2$?

Let's look at a slightly-modified function to gain some insight:

$$g(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

This function is also uniformly continuous. When we apply the limit quotient rule to $g(x)$ as x approaches 2, we again get:

$$\lim_{x\to 2} g(x) = \frac{0}{0}$$

But this time, the continuity of $g(x)$ leads us to conclude:

$$\frac{0}{0} = 4$$

The fault lies with the result of our limit. The quotient rule of limits is a valid operation to perform, but the result of that operation a meaningless limit. The statement that $\frac{0}{0} = 2$ is technically correct—by multiplying both sides by 0, we get $0 = 2 \cdot 0$. But by the same logic, the nature of $g(x)$ also shows that $\frac{0}{0} = 4$. The expression $\frac{0}{0}$ can represent any value, and is thus useless as a descriptor of a quantity. This is why it is often referred to as being "indeterminate".

In the context of this proof, the correct approach is to avoid using the limit quotient rule, since it does not give a meaningful result. Instead we should evaluate the limit directly:

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2-1}{x-1} \\ &= \lim_{x\to 1}(x+1) \\ &= 2\end{aligned}$$

We arrive at the much more exciting conclusion that $2 = 2$.

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